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# Production and Distribution in Political-Economic Systems: a Non-Atomic Game

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# Production and distribution in political-economic systems: a non-atomic game

[Alex Coram. Economics. University of Massachusetts. Draft 3. September 017]

## Abstract

*Societies are often compared in terms of the relative shares of the domestic product controlled by government but we don't have much by way of analytical tools to allow us to think about the ways in which changes in these shares affect the characteristics of production and distribution. This paper treats production and distribution as a cooperative game with a continuum of players. Outcomes depend on politics and economics. It has a number of unexpected results. Among these are that relative shares do not depend on the majority rule the production function and that the characteristics of inegalitarian and egalitarian systems may converge.*

## 1 Introduction

Societies are often compared in terms of the relative shares of the total domestic product transferred through government action and there is extensive discussion of the effects of differences in these shares on things like output, economic participation and distribution. Obvious examples are debates about welfare payments and a universal income. So far this discussion has mostly been historical and empirical. Despite the progress made with these approaches it would also be useful if it were possible to construct some sort of analytical framework that would help understand the factors influencing these differences and their affects at a more abstract level. This might help serve as a reference point for discussion in the same way as abstract models of, say, economic growth help us talk about particular economies. For example we might want to ask: What would be the properties of an allocation if every agent had a share based on political power and productive capacities? How does participation in production change as the share of distribution controlled by the political system changes? What happens as political power and the allocation of resources changes?

The purpose of this paper is to attempt to contribute to constructing at least one version of such a framework. Rather than starting with micro-foundations it looks at a distribution across large systems of political and economic interactions. I also undertake some comparative static analysis of systems with different levels of transfer.

The approach taken here is to treat the production and distribution problem as a cooperative game with an arbitrarily large number of agents thought of as a continuum. In contrast, most of the current theoretical literature on allocation through the political process concentrates on competitions between candidates or parties that offer resources to voters. Individual voters respond by trying to maximize their payoffs. This is usually done by making standard assumptions about utility functions and information and restricting the number of coalitions that form.<sup>1</sup> These restrictions strengthen the analysis in some directions by allowing the tools of optimization theory to be used, but they weaken it in others. In particular they leave out the macro-effects produced by the possibility of shifting sets of possible coalitions. For example with 30 voters and a majoritarian voting rule there are about 150 million possibilities.

It might be objected that the decision to ignore micro-foundations and use a model based on the value of the game is inconsistent with much contemporary practice in economics. This misses the point that the questions I am asking are different from micro-foundations questions. It also assumes that micro-foundations models are on a sound basis, or at least on a better basis than the approach used here.<sup>2</sup>

Many of the theoretical foundations for this paper have been developed by Aumann and Shapley [1], Aumann and Kurz [3], Shapiro and Shapley [17] and Milnor and Shapely [10]. In particular the second and third sections of the paper relies heavily on results in [1].

In the conclusion I summarise the characteristics of the systems studied and provide some answers to the questions in the first paragraph. Some results are reasonably obvious after the event. Others are less intuitive. One example is that the proportion of votes required to make a winning coalition and the nature of the production function does not effect the relative distribution. Another is that participation in production is similar in systems with equal distribution of endowments and systems with unequal distribution of political power.

I do not claim that these results map into the real world. On the other hand some of the results on the characteristics of systems with different shares of transfer are striking and, perhaps, should not be ignored in attempting to understand more specific cases.

I set out the paper as follows. In §.2 the problem and the model are presented. §.3 gives the main theorem. This is analyzed in §.4.

## 2 The model

### 2.1. Statement of the model

Suppose we have a social system made up of an economy and a political process. The problem presented

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<sup>1</sup>See Roemer [14] for an example of a two party model and good recent discussion of the literature on vote competition. Osborne and Slivinski [12] give a model where the number of candidates is not restricted but voter preferences are linear.

<sup>2</sup>There is a large literature which questions whether voters and economic agents behave in the way required for a micro-foundations approach. See for example Shapiro and Green [18].

in the introduction might be restated as that of determining how the total product would be distributed across all the members of a given system when everyone gets a share that is related to its political and economic capacities. The economy is described in terms of a production function, endowments of capital and labour capacity, and the political process represented by a voting mechanism. The total value for an individual is the sum of returns from participating in the political and production processes. The economy might be thought of as either capitalist in which each individual has an initial endowment of resources and skills, or a system in which all resources are collectively owned and workers have different skills. Although the model is written in terms of capitalism it holds for both cases. It is assumed that individuals do not extract rent and everyone has to work with their endowments. The system might be thought of as some sort of lemonade stand capitalism. There is full employment and everyone who wants to work has a job. The collective choice rule says that any coalition with a weight greater than half is winning. A winning coalition can distribute some portion of the produced goods to its members. This portion is fixed in the short run.

In many cases discussions of the effect of transfers from government on participation in the economy assume that there is a loss of utility associated with work. A similar assumption is made in this paper.

The size of a set, or coalition, of agents is defined by the Lebesgue measure  $\mu$  on the interval  $I = [0, 1]$ . This means  $\mu(I) = \inf \sum_i \nu_i = \sum_i \mu_i = 1$  where  $\nu_i$  is a sequence of intervals covering  $I$ . Let  $S$  be a coalition of agents. The measure of  $S$  is  $\mu(S) = \mu(S \cap I)$ . The majority rule is given by the number  $\theta$ . A coalition is winning when  $\mu(S) \geq \theta \geq \frac{1}{2}$ .

Each individual  $i$  is represented by an infinitesimally small sub-interval written  $dt$  or  $\mu(i)$  or  $\mu$  depending on the point to be made. To begin let  $\mu(i) = \mu(j)$  for all  $i, j$ . I will change this assumption later in the paper.

There is a single all purpose produced good and utilities in this good are linear. Utility of work is expressed in the same units as utilities of the produced good.

Individuals have different productive capacities depending on their endowments of resources and skills. Without worrying too much about measurement it is assumed that these endowments are substitutable. They can be thought of either as a vector or as a simple function  $g$  that takes on a finite number of values  $(g^1, \dots, g^k)$  on the interval  $I$ . Write the set of individuals with endowments  $g^r$  as  $G^r$  and think of  $g^r dt$  as the endowments of an individual in  $G^r$ . Since  $g$  is Lebesgue integrable define a vector

$$\gamma(S) := (\int_{G^1 \cap S} g^1 dt, \int_{G^2 \cap S} g^2 dt, \dots, \int_{G^k \cap S} g^k dt)$$

with  $\gamma^r(S) := \int_{G^r \cap S} g^r dt$ . It will be useful to express endowments as a measure by defining

$$w^r(S) := \frac{\gamma^r(S)}{\sum_j \gamma^j(I)}$$

where  $j = (1, \dots, k)$ . This means that  $w^r(S)$  is the endowments of type  $r$  in  $S$  expressed as a proportion of the total. It follows that

$$\sum_r w^r(I) = 1 = \mu(I) \quad (1)$$

It is assumed, for the time being, that  $w$  is a non-trivial partition of  $I$ . In other words there is no  $r : w^r(I) = 1$  and  $w^j(I) = 0$  for  $j \neq r$ .

It is assumed that the loss of utility from participation in production also affects utility from transfers. The obvious linkage is through loss of leisure. Loss of utility for agent  $i$  from participating in production is given by  $\bar{\epsilon}_i$  and from transfers is given by  $\epsilon_i$ . Let  $\epsilon_i = \epsilon_j = \epsilon$ .

The production function is assumed to be real valued, continuously differentiable, monotonically increasing and non-concave in  $w$ . In the comparative static analysis the value of  $a$  will influence  $w$  in a manner determined below. Hence

$$f(w, a) : \mathbf{R}^{k+1} \rightarrow \mathbf{R}$$

with  $f(0, a) = 0$ .

It is assumed that  $f$  can be expressed as a polynomial. This causes no loss of generality since the polynomials are dense in the space of continuously differentiable functions. See [4].

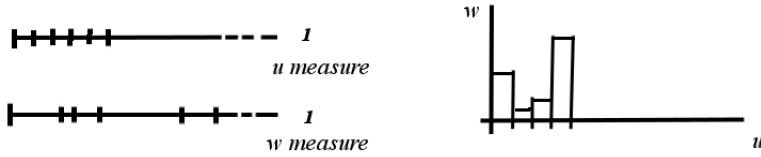


Figure 1. Measures on  $I$  and the construction  $w = w(\mu)$

Define the payoff to a coalition by monotonic non-decreasing set function. This function is super-additive and gives the maximum that  $S$  can achieve under the decision rules and the production technology

$$v : S \rightarrow \mathbf{R}$$

where  $v(\phi) = 0$ . It is assumed that  $v(S)$  is given in transferable utilities.

Write the share of the product that can be distributed by the winning coalition as  $a \in [0, 1]$ .

It is also assumed that agents are anonymous in the sense that if we perturb the set of agents the value of the game remains unchanged. Roughly speaking the name of the agents doesn't matter. This seems fairly unobjectionable in the present setting. I will modify it subsequently.

Let the game  $v$  be the sum of the political and economic games  $v = p + m$  and let  $N$  be the set of agents that participate in  $v$ . Then for any  $S \subset N$  the characteristic function is defined as

$$v(S) = \begin{cases} a(\epsilon f(w(I, a)) + (1 - a)(\bar{\epsilon} f(w(S, a))) & \text{when } \mu(S) > \theta \\ (1 - a)(\bar{\epsilon} f(w(S))) & \text{otherwise} \end{cases} \quad (2)$$

It might be worthwhile to bring three of the previous assumptions together for later reference. Let  $\varphi_i[v]$  be the value of a game  $v$ . Then: Ai. If  $v = p + r$  where  $p$  and  $r$  are games  $\varphi_i[v] = \varphi_i[p] + \varphi_i[r]$ . Aii.  $\sum_i \varphi_i = v(N)$ . Aiii. Anonymity.

## 2.2 The value for the game

The Aumann Shapely value generalizes the Shapley value and is the only suitable candidate for our purposes because it is the only function that satisfies Ai - Aiii ([19], p.31) with appropriate modifications for the continuous case ([11], p, 346).

The Shapley value is defined for an agent  $i$  as

$$\bar{\varphi}_i[v] := E\left[\sum_{S \ni i} (v(S \cup i) - v(S))\right] \quad (3)$$

where  $E$  is an expectations operator. This is given by the probability that agent  $i$  joins a coalition which already contains  $S$  agents when all orders are assigned equal probability.

It will be observed that since  $v(S \cup i) - v(S) > v(T \cup i) - v(T)$  for  $T \subseteq S$  the game is supermodular. This means that the Shapley value is in the core.

In order to extend the value defined over discrete sets to a continuum of agents it is necessary to show that it converges.

## 2.3 Convergence

The foundations for a proof of convergence are available in [1].<sup>3</sup>

**Theorem 1** *The value of the finite game  $v$  converges to the unique value of the continuous game if  $v$  can be expressed in linear powers of non-atomic measures.*

**Proof:** See Aumann and Shapley ([1], p.23) and [3] proposition 12.8.

□

To show that this theorem applies we need to prove the following proposition.

**Proposition 1:** *The production function  $f(w)$  can be written in linear powers of measures.*

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<sup>3</sup>See also Aumann and Kurtz [2, 3], Shapiro and Shapley [17] and Owen [11].

**Proof of lemma 1 :** Since  $w$  is a measure in  $\mathbf{R}^+$  for any fixed  $n$  and the product and sum of measures is a measure we only have to show that  $f(w)$  can be expressed in linear powers of products and sums of measures. It follows from the fact that  $f$  is a polynomial that the only terms which are not in powers of measures are of the form  $w_1^i w_2^j w_3^k \dots w_m^z$ . It will be seen that  $w_1^i w_2^j$  can be expressed in the form  $\frac{1}{2}(w_1^i + w_2^j)^2 - \frac{1}{2}(w_1^{2i} + w_2^{2j})$ . It follows by induction that  $w_1^i w_2^j w_3^k \dots w_m^z$  can be expressed as powers of partial sums and powers.

□

## 2.4. Rewriting the production function

It is possible to make life easier by simplifying the production function, This is done using the idea that, if a coalition is sufficiently large, the proportion of different sorts of agents in its composition would be expected to approximate the proportion in the population as a whole.

**Proposition 2.** For a coalition of measure  $\mu(S) = t$  there is an  $n$  sufficiently large that the probability  $\{|w(S) - t(w(I))| \geq c\} \leq \delta$  for any  $\delta > 0$ .

**Proof.** This is a straightforward application of Tchebychev's inequality and since we are dealing with a continuum of agents we are done.

□

It follows from proposition 1 that we can rewrite the production function for a coalition of measure  $\mu(S) = t$  as

$$f(\omega(S, a)) = f(t\omega(I, a)) + o \quad (4)$$

where  $o$  is an error term that is vanishingly small.

In what follows the main theorem is stated with  $\epsilon := 1, \bar{\epsilon} := 1$  and this is called the basic model. The relative endowments of individual  $i$  are  $w^r(i)$ . To simplify these are usually called endowments and written  $w^r$  where there is no ambiguity.

## 3 The value theorem

### 3.1. The main theorem for the basic model

The main theorem can be interpreted in an after the event manner in terms of averaging across returns to contribution. This is a remarkable result because the initial assumptions are more primitive and do not contain any reference to this process. As an example of this idea, if a market were sufficiently large and factors of production were sufficiently mobile and we wanted to understand returns it might simply be assumed that each factor gets its average contribution. It might also be reasonable to assume that, if a group of voters is ignored in one election it is likely that some attempt will be made to buy its votes in subsequent elections. Over time an average might again be a first approximation before any details are added. It also says that the value for an agent is determined by its political power and its share of endowments. Relative shares are

independent of the majority rule and the production function.

### 3.2. Main theorem

**Theorem 2** . *The value for an individual  $i \in N$  with endowments  $w^r$  and  $\epsilon = \bar{\epsilon} = 1$  is given by*

$$\varphi_i[v] = f(w(I, a))(a\mu + (1 - a)w^r)$$

This result can be proven by rewriting equation (2) in terms of equation (4) and using this to construct the expected value. Here is a more slick proof.

**Proof of Theorem 2.** Since the Shapley value is the unique function that satisfies Ai - Aiii all that is required is to show that  $\varphi_i$  satisfies these conditions. This is immediate.

□

It is now time to support the assertions made at the beginning of this section. This is done in the following corollary.

**Corollary 1 of Theorem 2.** [a]. The distribution is independent of  $\theta$  and relative distribution is independent of  $f$ ;

[b]. (i). An increase in  $a$  benefits  $i$  if  $w^r(i) < \mu$  and harms  $i$  otherwise; (ii).  $v(S)_a = 0$ .

**Proof.** [a]. The parameter  $\theta$  does not appear in  $\varphi_i$  and independence of relative shares from  $f$  is obvious. [b]. (i). Differentiating  $\varphi_i$  with respect to  $a$  gives the result; (ii). For a coalition the value becomes  $\varphi[v(S)]$ . The value of a coalition is the sum of the value of its members so  $\varphi[v(S)]$  is  $a f \sum \mu(S)$  and  $(1 - a) \sum w^r(S)$ . Since  $\sum \mu(S) = \sum w^r(S)$  the  $a$  disappears as required.

□

In addition to supporting the claims that shares not depend  $\theta$  and relative shares do not depend on  $f$  this shows that an increase in transfers benefits agents with less than average productivity and harms them otherwise. It also shows that a change in the proportion of the total product subject to transfers does not affect  $v(S)$ .

It is easy to prove these results but they were not necessarily obvious from the specification of the game. It might have been thought, for example that making the size of the coalition required to win would increase the value of less productive players or that relative shares would depend on the production function.

## 4 Comparative statics for a smooth approximation

### 4.1. The smooth approximation

Theorem 2 is now used to make a static comparison of systems under different values of the parameter  $a$



when  $\epsilon, \bar{\epsilon}$  are not identically 1. It is assumed that in each system agents are partitioned into different games depending on whether the value of participating in production is positive or negative. There is no dynamic analysis of how this partition occurs. Maybe it represents some long run historical re-alignment of different systems in which agents have shifted to the coalitions that give them the greatest expected payoffs over time.

Let  $\epsilon_i = \bar{\epsilon}_i \leq 1$  when an agent works. When an agent does not work  $(\epsilon, \bar{\epsilon}) = (0, 1)$ .

It is assumed that agents are distributed between a game with carrier  $N$  and a game with carrier  $M$  where  $I = N \cup M$  depending on which game has the greater value for each agent. Let  $N$  be the carrier for  $v = p + r$  and  $M$  the carrier for  $u = p + q$  with  $N \cap M = \emptyset$ . Let  $q(\hat{S}) = (1 - \epsilon)p(\hat{S})$  for any  $\hat{S} \subset M$ . Then we have  $u(\hat{S}) = (p + q)(\hat{S})$  given by

$$u(\hat{S}) = \begin{cases} af \circ (w(I, a)) & \text{when } \mu(S \cup \hat{S}) > \theta \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

**Lemma** The games  $v$  and  $u$  are additive for all  $i \in N \cup M$  and satisfy anonymity.

**Proof.** For  $i \in N$  we have  $\varphi_i[v + u] = \varphi_i[p + r]$  with  $\varphi_i[q] = 0$  and for  $j \in M$  we have  $\varphi_j[v + u] = \varphi_j[p + q]$  with  $\varphi_j[r] = 0$ . It follows that  $\varphi_i[v + u] = \varphi_i[p + r + q] = \varphi_i[p] + \varphi_i[r] + \varphi_i[m]$ . It is obvious that anonymity holds for  $S \cup \hat{S}$  in the game  $p$  and in the games  $r$  and  $q$ .

In what follows the analysis is simplified by constructing a smooth approximation to the game. This is done by constructing the  $w$  measures as a differentiable function  $W : t \rightarrow \mathbf{R}$ . Issues of existence and uniqueness for  $W$  are avoided because the approximation is for  $\varphi$  and this is determined on the space of measures.

To construct the approximation rank the elements of  $w$  and treat the result as a step function on  $[0, 1]$  that is monotonically non-decreasing in  $t$ . Interpolate this with a smooth function  $W$ . To save notation the interpolating function is rewritten  $w(t)$ . An agent with capacity  $r$  will be assumed to be located at  $t = r$  for  $r$  suitably scaled such that  $r \in [0, 1]$  and  $w^r = w(r)$ . This preserves continuity with the discrete endowments but the meaning of  $r$  should not be confused.

In what follows the smooth approximation is used to move between the two interpretations of  $w$ . It takes advantage of the fact that if  $w(c) > w(r)$  is needed to satisfy some parameter values then a capacity  $w^c \geq w^r$  will be required to satisfy the same values.

If the returns for working are greater than the returns from not working agents  $i \in S$  with capacity  $w(c) > w(r)$  then  $S \in N$  the set of working agents. Because the value of  $a$  will determine the partition  $N, M$ , it will also determine  $w(I, a)$ . If an increase in  $a$  causes  $M$  to increase  $f$  changes from  $f(w(I, a))$  to  $f(w(I, a) - \delta w(I, a))$ .

An agent will be allocated to  $N$  if

$$s_i = a\epsilon f(w(I, a))\mu - af(w(I, a) - \delta w(I, a))\mu + (1 - a)(\epsilon f(w(I, a))w(r)) > 0$$

and otherwise to  $M$ .

Define  $\beta(a) : \beta f(w(I, a)) = f(w(I, a) - \delta w(I, a))$ . Rearranging

$$s_i = f(a\mu(\epsilon - \beta) + (1 - a)\epsilon w(r)) \quad (6)$$

with  $s_i > 0$  for  $i \in N$ .

#### 4.2. Results for the smooth system

The set  $M$  of non-producing agents will depends on distribution of productive capacities. Consider two systems with the same capacity to produce. In this case  $M$  will be larger at lower levels of transfer in the system with the less egalitarian initial distribution. Beyond some point, however, the set of non-working agents will increase much more rapidly in the system with the more egalitarian distribution.

One way to think about this is in terms of endowments rather than agents. Let  $n = |N|$  and  $m = |M|$  measured in endowments. This means  $m = 1$  when  $w(r) \in M$  for  $r = 1$  which is the same as  $\sum_0^k w^r = 1$  with discrete endowments. It follows  $n = 1 - m$ .

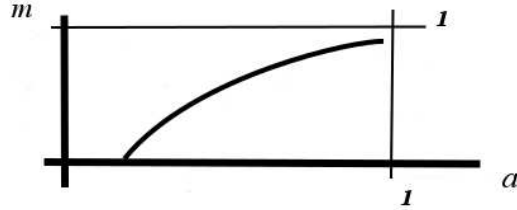


Figure 2. Example of  $m(a)$

In order to consider changes in the parameter  $a$  rewrite equation (6). The condition required for agents with some level of endowments  $w(r)$  to be in  $N$  is that  $w(r) > \inf w$  where  $\inf w = \bar{w}$  is the smallest  $w$  such that  $s_i = 0$ . This is

$$\bar{w} = \frac{a\mu(\beta - \epsilon)}{(1 - a)\epsilon} \quad (7)$$

and it follows that  $\bar{w} > 0$  for  $a > 0$  since  $\beta(0) = 1 > \epsilon$ .

**Corollary 2 of Theorem 2.** [a] Suppose  $a$  increases. Then  $\bar{w}$  increases until  $w = w(c)$  for some  $t = c < 1$  with  $m < 1$ . For  $a = 1$  the system is indeterminate.

[b] For  $a$  sufficiently small  $\bar{w} < \mu$ .

**Proof.** [a]. Differentiating equation (7)

$$\bar{w}_a = \frac{\mu(a(1-a)\beta_a + \beta - \epsilon)}{((1-a)^2\epsilon)} \geq 0$$

To show  $\bar{w}_a \geq 0$  assume  $\bar{w}_a < 0$ . In this case  $\beta_a > 0$ . This means  $\bar{w}_a \geq 0$ . Now assume  $\bar{w} \rightarrow \max w = w(1)$  for  $a < 1$ . In this case  $\beta \rightarrow 0$  and for  $\epsilon = z > 0$  where  $z$  is fixed there is some  $\bar{w}(r)$  for  $r < 1 : \beta - \epsilon < \delta$  where  $\delta > 0$  is sufficiently small that  $\bar{w}(r)_a < 0$ . Contradiction.

[b] Obvious from equation (6).

□

In crude terms [a] says that as  $a \rightarrow 1$  and  $\beta$  goes to some value less than  $\epsilon + \delta$  from above the rate at which  $r : w(r) = \bar{w}$  increases goes to zero. Since the partition is non-trivial there must be agents with capacity  $w^t > w^r$  in  $N$  and hence  $m < 1$ . In other words  $N \neq \phi$  in every system for  $a < 1$ . When  $a = 1$  the outcome is indeterminate.

#### 4.3. Variations in endowments

The endowments of individuals are now varied by making the distribution more equal. In this case the range over which  $m$  increases in the non-trivial partition shrinks towards a point with very little withdrawn one side and nearly everything withdrawn the other. To see these results construct the homotopy

$$H = [\alpha^h, t] : w \rightarrow \mu$$

for  $h \in [0, 1]$  with  $\alpha^0(t) = w(t)$  to  $\alpha^1(t) = \mu$ . For every  $h$   $\alpha^h$  must satisfy  $\int_0^1 \alpha^h(t) dt = c$  for  $c > 0$  a constant chosen to preserve the condition  $\sum_I \mu = \sum_I w^r = 1$ . Write the differential of  $\alpha^h$  with respect to  $h$  at  $t$  as  $\alpha_h^h$ .

**Corollary 2 of Theorem 2 continued.** The changes in  $r : w(r) = \bar{w}$  for  $H : w \rightarrow \mu$  are: [c] For  $a : \bar{w} < \mu$  we have  $r \rightarrow 0$  continuously.

[d] For  $a : \bar{w} > \mu$  we have  $r \rightarrow 1$  continuously.

**Proof.** [c]. Fix  $\beta$ . Since  $\mu$  remains constant  $\bar{w}$  is constant in equation (7) and the value of  $r : w(r) = \bar{w}$  must change. For  $\bar{w} < \mu$  we have  $\alpha_h^h > 0$  for all  $t : w(t) < \mu$  and  $w$  is increasing at each  $t$ . It follows that  $r$  must decrease to  $r' < r$  to set  $w(r') = \bar{w}$ . Now let  $\beta$  vary. This means  $\beta$  increases because less productive capacity is withdrawn but the increment  $\delta\beta$  can only partially offset the reduction in  $r$ . If not  $\beta$  would increase. Continuity. All we need is that  $s_i$  is continuous in  $\beta$  and  $w$ . For any  $a : \bar{w} < \mu$  there is an  $h : \alpha^h$  is sufficiently close to  $\mu$  that  $r : w(r) = \bar{w} < \delta$  for any  $\delta > 0$ .

[d]. In this case  $\alpha_h^h < 0$  for all  $t : w(t) > \mu$  and the proof is similar. Continuity. For  $h : \alpha^h$  sufficiently close to  $\mu$  we have  $r > 1 - \delta$  for any  $\delta > 0$ . Note from 2 [a] that  $r \neq 1$  for  $a < 1$ .

□

An illustration of the proof is given in fig. 3.

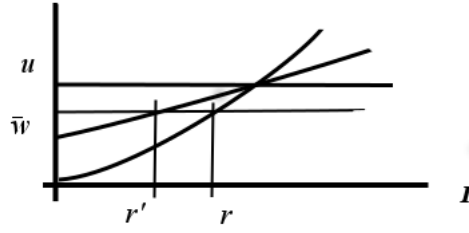


Figure 3. Changes in  $r$  as  $w \rightarrow \mu$ .

If the constraint that  $w \neq \mu$  is removed we can look at the limit

$$s_i = f(\epsilon - a\beta)\mu$$

and it follows that the sign on  $s_i$  does not depend on  $w$  or  $\mu$ . This means that for  $a$  sufficiently small all agents with  $w(r) > 0$  work. For  $a : a\beta \geq \epsilon$  the system becomes indeterminate.

#### 4.4. Variations in political power

The political power of individuals may vary as the result of exogenous influences including the ability to use wealth to influence the outcome of elections or to gain favourable transfers. There are a number of ways to think about this. Here are two examples.

**Example 1. Transfers and wealth** The returns from the transfer process may depend on wealth for two reasons. The first is that transfers to individuals with equal voting power might depend on endowments and the second is that the probability of being decisive may be a function of economic capacity. The first case is more difficult to deal with but it seems more natural to treat everyone as having a vote and to let the payoffs vary. This is referred to in the literature as a weighted Shapley value. A downside of doing this is that the anonymity condition is lost.

It needs to be shown that it is possible to construct a homotopy that takes  $\mu$  to  $w(r)$  where  $\mu$  is now treated as a share of transfers.

**Corollary 2 of Theorem 1.** There is a homotopy  $G = [\alpha^g, t] : \mu \rightarrow w$  for  $g \in [0, 1]$  where  $\mu$  is a share of transfers.

**Proof.** Consider the discrete case for the game  $p$ . Define the weight of an individual in the interval around  $t = i$  in coalition  $S$  be  $\varpi_i = \frac{\lambda_i}{\lambda(S)}$  where  $\varpi_{iS} > 0$  and the  $\varpi$  are measures in  $[0, 1]$  with  $\varpi(N) = \sum_i \varpi_i = 1$ . It is known that the vector of unanimity games  $p_S$  forms a basis for the set of all games  $p$  [19], ([11] p. 263). This means that the weighted payoff can be defined as

$$\varphi_{i\varpi}[p] := \lambda_i \sum_{S \subset N} \frac{c_S}{\lambda(S)} \varphi_i[p_S]$$

for  $c_S$  a real number associated with each coalition  $S$ . Theorem 1 says that the value of the finite game converges to the value of the continuous game if it can be expressed in powers of measures. Since  $\varphi_{i\varpi}[p]$  is a sum of the form  $\sum \varpi_{iS} c_S \varphi_i[p_S]$  for  $c_S \varphi_i[p_S] \in \mathbf{R}$  it can be expressed as a sum of measures. It follows that  $\varphi_{i\varpi}[p] = \xi(\varpi) \varphi_i[p] = \xi(\varpi) \mu f$ . It is now possible to use  $\xi(\varpi) \mu$  to define  $\alpha := \alpha(\frac{\mu}{\xi(\varpi)})$  with  $\alpha^0 = \mu$  and  $\alpha^1 = w(r)$ .

□

This gives the following results for the non-trivial partition.

**Corollary 3 of Theorem 2.** Let  $\mu \rightarrow w(t)$ . [a]. For  $\bar{w} < \mu$  we have  $\bar{w} \rightarrow 0$  and  $m \rightarrow 0$  continuously. [b]. For  $\bar{w} > \mu$  we have  $\bar{w} \rightarrow w \max$  and  $m \rightarrow 1$  continuously.

**Proof.** [a]. Rewriting equation (7)

$$\bar{w} = \frac{a\alpha^g \mu(\beta - \epsilon)}{(1-a)\epsilon}$$

Hold  $\beta$  constant. Differentiating gives  $\bar{w}_g = \frac{a\alpha_g^g \mu(\beta - \epsilon)}{(1-a)\epsilon}$  and since  $\alpha_g^g < 0$  we have  $\bar{w}_g < 0$  and  $m$  decreasing. Now let  $\beta$  vary with  $\beta_g > 0$  because  $m_g < 0$ . This means that  $\beta$  is increasing to offset the decrease from  $\alpha^g$ . If this increase is sufficient to make  $\bar{w}_g > 0$  then  $\beta$  is increasing. Contradiction. Hence  $m \rightarrow 0$  as required. Continuity is obvious.

[b] This is similar to the previous with  $\bar{w}_g > 0$ . This means  $m \rightarrow 1$ . Note from 2 [a] that  $c \neq 1$  for  $a < 1$ .

□

An example is given in fig. 4.

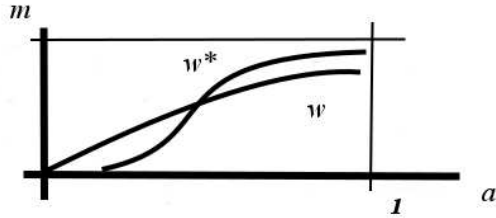


Figure 4. Example of  $m$  for  $|w * - \mu| < |w - \mu|$  for any  $t$ .

If we relax the constraint that  $\mu = w(t)$

$$s_i = f(\epsilon - a\beta)w$$

and it follows that the sign on  $s_i$  does not depend on  $w$ . For  $a$  sufficiently small all agents with  $w(r) > 0$  are in  $N$ . For  $a : \epsilon \geq a\beta$  the system becomes indeterminate.

If this result is compared with the result in corollary 2 [d] and [e] we get the surprise that the characteristics of the systems are the same when  $w = \mu$  regardless of whether the adjustment is in  $w$  or in  $\mu$ . A complication is added by the discontinuity at the limit  $w = \mu$ . However in both cases they converge on the limit of the non-trivial partition continuously. In addition for  $\epsilon > a\beta$  and  $a\beta \rightarrow \epsilon$  they approach the limit  $\lim_{\mu \rightarrow w}$  continuously.

In retrospect it is possible to construct some sort intuitive explanation for all this. However, for me at least, it was not obvious at the outset that  $m$  and hence  $f$  would converge in a system in which endowments became increasingly equal and a system in which political power became increasingly unequal. It seemed ore reasonable to expect that such differences would produce different results.

**Example 2. Major player.** It might be the case that political and economic power is concentrated in the hands of a small number of major players. Maybe the major player has a disproportionate return from the political process or a large proportion of endowments. To take the simplest example assume that a single large player has a probability  $c$  of determining the election result and  $bc$  of the economic resource for  $bc \in (0, 1)$ . This gives endowments for each unit of political power as  $b$ . This gives an oceanic game [10] in which the value for a minor player becomes

$$\varphi_i = f \circ (\psi w(I))((1 - c)\epsilon a\mu + (1 - bc)\bar{\epsilon}(1 - a)w(r))$$

In this case the ratio  $b$  is the important determinant of the value for the minor players. This is shown in the following.

**Corollary 3 of Theorem 2 continued.**[b]. (i). Set  $\epsilon = \bar{\epsilon} = 1$ . The value of  $w(r)$  such that  $\varphi_i$  increases as  $a$  increases becomes larger at an accelerating rate as  $b$  increases. (ii) For  $b > 1$  this value is  $w > \mu$ .

[c] For  $\epsilon < 1$  and  $\bar{\epsilon} = (1, 0)$  we have  $\bar{w} : s_i = 0$  increasing at an increasing rate as  $b$  increases.

**Proof** [b](i).  $\varphi_{ia} = 0$  when  $w(t) = \frac{(1-c)\mu}{(1-bc)}$ . Differentiating gives the results. (ii). Immediate.

[c].  $\bar{w} = \frac{(1-c)(\beta-\epsilon)a\mu}{(1-bc)(1-a)\epsilon}$ . As before differentiation gives the result.

In other words in a system where the ration of endowments to political power of the major player becomes greater the benefits from transfers are more widely spread and participation falls. Interestingly if  $b = 1$  nothing changes. That is the value of  $w(r)$  at which agents benefit from transfers and the value of  $\bar{w}$  remains the same for all concentrations of endowments and political power.

## 5 Conclusion

The general characteristics of systems with different levels of transfers are as follows.

[i]. The level of inequality in shares of production decreases as transfers increase if  $w \neq \mu$ .

[ii]. The majoritarian rule does not affect the final distribution and the production function does not affect relative shares of the total product.

In the case where working causes a loss of utility:

[iii]. The proportion of agents in the non-producing coalition  $M$  for each level of  $a$  will depend on the relation between  $\mu$  and  $w$ .

(a). If  $w \neq \mu$  some individuals participate in production in all systems until  $a = 1$  and the game becomes indeterminate.

(b). As  $w \leftrightarrow \mu$  there is convergence between the system with increasing equality in distribution and the system with increasing inequality in political power.

(c). For  $w = \mu$  all agents produce until the level of transfers reaches a critical point.

[iv]. If there is a major player and the relative share of its endowments to its political power increase :

(a) the returns to transfers relative to the returns to endowments of the minor players increases;

(b) the number of agents in  $M$  increases.

Even though I make no claims about reality it might be possible to translate some of these results into observable outcomes and subject them to testing. At the least it should be possible to incorporate them into explanations of differences between systems such as the US and some European countries.

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